PARAMETRIC METHOD OF CALCULATING AN UNSTEADY LAMINAR BOUNDARY LAYER IN AN INCOMPRESSIBLE LIQUID WITH SUCTION OR INJECTION

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The equation for the unsteady boundary layer at a porous wall is reduced with the help of three series of parameters to a universal form not containing explicitly either the velocity at the outer edge of the boundary layer or the velocity of suction or injection.

1. A generalization of the parametric method of Loitsyanskii [1] to the case of a steady laminar boundary layer at a porous wall was performed by Chan [2]. In the present article a study is made of the unsteady laminar boundary layer with suction or injection, where the velocity v_W of suction or injection can be assigned in the general case as a function of time and of the longitudinal coordinate.

In analyzing the excess transverse velocity component $v_1 = v - v_W$ in the boundary layer, we introduce the stream function ψ as follows:

$$u = \frac{\partial\psi}{\partial y}, \quad v_1 = -\frac{\partial\psi}{\partial x}.$$

Then the equation for the unsteady laminar boundary layer in an incompressible liquid in the presence of suction or injection will be

$$\frac{\partial^2\psi}{\partial t\partial y} + \frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\cdot\frac{\partial^2\psi}{\partial y^2} + v_w\frac{\partial^2\psi}{\partial y^2} = UU' + \dot{U} + \nu\frac{\partial^3\psi}{\partial y^3} \tag{1}$$

with the boundary and initial conditions

$$\left.\begin{aligned} &\psi = \frac{\partial\psi}{\partial y} = 0 \quad \text{at} \quad y = 0, \quad \frac{\partial\psi}{\partial y} \to U(x, t) \quad \text{as} \quad y \to \infty, \\ &\frac{\partial\psi}{\partial y} = u_1(x, y) \quad \text{at} \quad t = t_0, \quad \frac{\partial\psi}{\partial y} = u_0(t, y) \quad \text{at} \quad x = x_0 \end{aligned}\right\} \tag{2}$$

(a partial derivative with respect to the x coordinate is denoted by a prime and a partial derivative with respect to time t is denoted by a dot).

We introduce the new variables

$$x = x; \quad t = t; \quad \eta = \frac{By}{h(x, t)}; \quad \varphi(x, \eta, t) = \frac{B\psi(x, y, t)}{U(x, t)\,h(x, t)}, \tag{3}$$

where h(x, t) is some as yet arbitrary characteristic linear scale of the transverse coordinate in the boundary layer and B is a normalizing constant. Designating $z = h^2/\nu$, we can reduce Eq. (1) to the form

$$B^2\frac{\partial^3\varphi}{\partial\eta^3} + \left(\frac{\eta\dot{z}}{2} + \frac{Uz'}{2}\varphi + zU'\varphi - B\frac{hv_w}{\nu}\right)\frac{\partial^2\varphi}{\partial\eta^2}$$

$$+ zU'\left[1 - \left(\frac{\partial\varphi}{\partial\eta}\right)^2\right] + \frac{\dot{U}}{U}z\left(1 - \frac{\partial\varphi}{\partial\eta}\right) = zU\left[\frac{\partial\varphi}{\partial\eta}\cdot\frac{\partial^2\varphi}{\partial\eta\partial x} - \frac{\partial\varphi}{\partial x}\cdot\frac{\partial^2\varphi}{\partial\eta^2}\right] + z\frac{\partial^2\varphi}{\partial\eta\partial t} \tag{4}$$

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with the boundary conditions

$$\varphi = \frac{\partial\varphi}{\partial\eta} = 0 \quad \text{at} \quad \eta = 0, \quad \frac{\partial\varphi}{\partial\eta} \to 1 \quad \text{as} \quad \eta \to \infty. \tag{5}$$

The conditions written in the last line of (2), as usual in the derivation of a universal equation, are not taken into account but only used in the solutions of particular problems. To universalize Eq. (4) we introduce into the analysis a multiparametric family of velocity profiles in cross sections of the boundary layer in the form

$$\frac{u}{U} = \frac{u}{U}(\eta, f_{kn}, g_{lm}, \lambda_{ij})$$

and the corresponding stream function

$$\psi = \frac{Uh}{B}\varphi(\eta, f_{kn}, g_{lm}, \lambda_{ij}), \quad k, n, l, m, i, j = 0, 1, 2, \ldots, \tag{6}$$

where the three series of parameters are written as follows [we assume that the functions $U(x, t)$, $v_w(x, t)$, and $z(x, t)$ are analytical]:

$$f_{kn} = U^{k-1}\frac{\partial^{k+n}U}{\partial x^k \partial t^n} z^{k+n}, \tag{7}$$

$$g_{lm} = U^l \frac{\partial^{l+m}z}{\partial x^l \partial t^m} z^{l+m-1}, \tag{8}$$

$$\lambda_{ij} = -U^i \frac{\partial^{i+j}v_w}{\partial x^i \partial t^j} \cdot \frac{z^{i+j+\frac{1}{2}}}{\sqrt{\nu}}. \tag{9}$$

These parameters, which substitute for the longitudinal coordinate x and the time t, reflect the effect on the characteristics of the boundary layer of the velocity in the outer stream and the velocity of suction or injection, as well as the past history of flow in the boundary layer (through the quantity z and its derivatives). With arbitrary functions of the velocity at the outer limit of the boundary layer and the velocity of suction or injection and with an arbitrary scale of the transverse coordinate all the parameters are independent of one another and are henceforth considered as independent variables. Let us write the values of the parameters with the initial indices. From (7)-(9) we will have

$$\begin{gathered} f_{10} = U'z, \quad f_{01} = \frac{\dot{U}}{U}z, \quad g_{10} = Uz', \quad g_{01} = \dot{z}, \\ \lambda_{00} = -v_w\frac{\sqrt{z}}{\sqrt{\nu}}, \quad \lambda_{10} = -Uv_w' z\frac{\sqrt{z}}{\sqrt{\nu}}, \quad \lambda_{01} = -\dot{v}_w z\frac{\sqrt{z}}{\sqrt{\nu}}. \end{gathered} \tag{10}$$

The parameters $f_{00} = 1$ and $g_{00} = 1$ as constants are not taken into account. We note that the single parameter $g_{01} = \dot{z}$, which is one of the parameters of (8), was used in [3]. But the introduction of the full series of parameters g_{lm}, including the derivatives of the scale z with respect to both x and t, allows one to make more detailed allowance for the past history of flow in the boundary layer and thus to introduce generalization into the statement of the problem. We note that in the case of steady motion of the liquid in the boundary layer without suction or injection one can use simplified series (7) and (8) in the form

$$f_k = U^{k-1}\frac{\partial^k U}{\partial x^k}z^k, \quad g_l = U^l\frac{\partial z^l}{\partial x^l}z^{l-1}.$$

We then write the derivatives with respect to the longitudinal coordinate and time as follows:

$$\begin{gathered} \frac{\partial}{\partial x} = \sum_{k,n=0}^{\infty}\frac{\partial}{\partial f_{kn}}\cdot\frac{\partial f_{kn}}{\partial x} + \sum_{l,m=0}^{\infty}\frac{\partial}{\partial g_{lm}}\cdot\frac{\partial g_{lm}}{\partial x} + \sum_{i,j=0}^{\infty}\frac{\partial}{\partial \lambda_{ij}}\cdot\frac{\partial \lambda_{ij}}{\partial x}, \\ \frac{\partial}{\partial t} = \sum_{k,n=0}^{\infty}\frac{\partial}{\partial f_{kn}}\cdot\frac{\partial f_{kn}}{\partial t} + \sum_{l,m=0}^{\infty}\frac{\partial}{\partial g_{lm}}\cdot\frac{\partial g_{lm}}{\partial t} + \sum_{i,j=0}^{\infty}\frac{\partial}{\partial \lambda_{ij}}\cdot\frac{\partial \lambda_{ij}}{\partial t}. \end{gathered} \tag{11}$$

The derivatives of the parameters with respect to the coordinates are found by direct differentiation of Eqs. (7)-(9). We have

$$
\begin{aligned}
\frac{\partial f_{kn}}{\partial x} &= \frac{D_{kn}(f_{kn}, g_{10})}{Uz}, \quad \frac{\partial f_{kn}}{\partial t} = \frac{E_{kn}(f_{kn}, g_{01})}{z}, \\
\frac{\partial g_{lm}}{\partial x} &= \frac{K_{lm}(g_{lm}, f_{10})}{Uz}, \quad \frac{\partial g_{lm}}{\partial t} = \frac{L_{lm}(g_{lm}, f_{01})}{z}, \\
\frac{\partial \lambda_{ij}}{\partial x} &= \frac{M_{ij}(\lambda_{ij}, f_{10}, g_{10})}{Uz}, \quad \frac{\partial \lambda_{ij}}{\partial t} = \frac{N_{ij}(\lambda_{ij}, f_{01}, g_{01})}{z},
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
D_{kn}(f_{kn}, g_{10}) &= (k-1) f_{10} f_{kn} + (k+n) f_{kn} g_{10} + f_{k+1,n}, \\
E_{kn}(f_{kn}, g_{01}) &= (k-1) f_{01} f_{kn} + (k+n) f_{kn} g_{01} + f_{k,n+1}, \\
K_{lm}(g_{lm}, f_{10}) &= l g_{lm} f_{10} + (l+m-1) g_{10} g_{lm} + g_{l+1,m}, \\
L_{lm}(g_{lm}, f_{01}) &= l g_{lm} f_{01} + (l+m-1) g_{01} g_{lm} + g_{l,m+1}, \\
M_{ij}(\lambda_{ij}, f_{10}, g_{10}) &= i f_{10} \lambda_{ij} + \left(i+j+\frac{1}{2}\right) g_{10} \lambda_{ij} + \lambda_{i+1,j}, \\
N_{ij}(\lambda_{ij}, f_{01}, g_{01}) &= i f_{01} \lambda_{ij} + \left(i+j+\frac{1}{2}\right) g_{01} \lambda_{ij} + \lambda_{i,j+1}.
\end{aligned}
\tag{13}
$$

Using Eqs. (10)-(13) we can convert Eq. (4) to the form

$$
\begin{aligned}
&B^2 \frac{\partial^3 \varphi}{\partial \eta^3} + \left(\frac{\eta g_{01}}{2} + \frac{g_{10}}{2}\varphi + f_{10}\varphi + B\lambda_{00}\right) \frac{\partial^2 \varphi}{\partial \eta^2} + f_{10}\left[1 - \left(\frac{\partial \varphi}{\partial \eta}\right)^2\right] \\
&+ f_{01}\left(1 - \frac{\partial \varphi}{\partial \eta}\right) = \sum_{k,n,l,m,i,j=0}^{\infty} \left[\left(\frac{\partial \varphi}{\partial \eta} \cdot \frac{\partial^2 \varphi}{\partial \eta \partial f_{kn}} - \frac{\partial \varphi}{\partial f_{kn}} \cdot \frac{\partial^2 \varphi}{\partial \eta^2}\right) D_{kn}\right. \\
&+ \frac{\partial^2 \varphi}{\partial \eta \partial f_{kn}} E_{kn} + \left(\frac{\partial \varphi}{\partial \eta} \cdot \frac{\partial^2 \varphi}{\partial \eta \partial g_{lm}} - \frac{\partial \varphi}{\partial g_{lm}} \cdot \frac{\partial^2 \varphi}{\partial \eta^2}\right) K_{lm} \\
&\left. + \frac{\partial^2 \varphi}{\partial \eta \partial g_{lm}} L_{lm} + \left(\frac{\partial \varphi}{\partial \eta} \cdot \frac{\partial^2 \varphi}{\partial \eta \partial \lambda_{ij}} - \frac{\partial \varphi}{\partial \lambda_{ij}} \cdot \frac{\partial^2 \varphi}{\partial \eta^2}\right) M_{ij} + \frac{\partial^2 \varphi}{\partial \eta \partial \lambda_{ij}} N_{ij}\right].
\end{aligned}
\tag{14}
$$

In the solution in general form the constant in the equation should be taken as B = 1. Then the boundary conditions for Eq. (14) will be

$$
\begin{aligned}
&\varphi = \frac{\partial \varphi}{\partial \eta} = 0 \ \text{at} \ \eta = 0, \quad \frac{\partial \varphi}{\partial \eta} \to 1 \ \text{as} \ \eta \to \infty, \\
&\varphi = \varphi_0(\eta) \ \text{at} \ g_{10} = 2, \ f_{kn} = g_{lm} = \lambda_{ij} = 0
\end{aligned}
\tag{15}
$$

(in the last equality the parameter g_{10} is excluded from the series g_{lm}); $\varphi_0(\eta)$ is the solution of the Blasius equation for the steady boundary layer at a plate. The equation and boundary conditions (14) and (15) are universal, since they do not contain in explicit form the velocity at the outer limit of the boundary layer or the velocity of suction or injection. The integration of the universal equation can be carried out as usual on a computer by "segments" once and for all for different values of the parameters under consideration. As a result of the integration one determines the velocity fields as well as the required characteristic functions of the boundary layer, particularly the reduced friction, which is calculated from the equation $\zeta(f_{kn}, g_{lm}, \lambda_{ij}) = \partial^2\varphi/\partial\eta^2 |_{\eta=0}$.

The "universal" functions obtained are used to solve a concrete problem with assigned distributions of U(x, t) and $v_w(x, t)$, although in doing this one must determine the scale h(x, t). The choice and the means of calculation of the characteristic scale, with observance of the condition $h \sim 1/\sqrt{\mathrm{Re}}$, can be different, generally speaking, although they have an important effect on the accuracy of solution of the problem. We note that the use for this purpose of the thickness δ^{**} of momentum loss or another quantity determined from some integral relation and "following," as it were, the development of the boundary layer along the longitudinal coordinate and in time usually leads to good results in terms of the rapidity of convergence of the method.

In the general statement of the problem under consideration the introduction of the additional series of parameters g_{lm}, as is seen, frees one from the use of integral equations in the derivation of the universal equation, i.e., in the first stage of the solution, and thereby allows one to obtain the universal equation in a

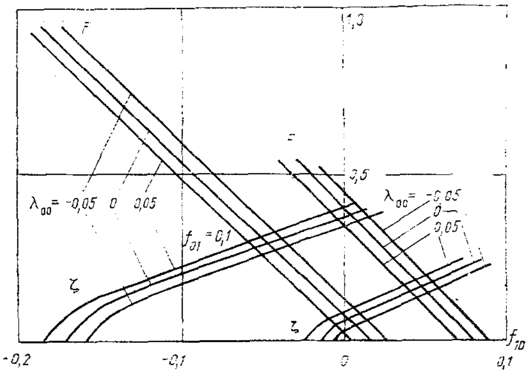

Fig. 1. Dependence of reduced coefficient of friction and of the functional F on the parameters f_{10} and λ_{00} with a constant value $g_{01} = 0.1$ for $f_{01} = 0.1$ and $f_{01} = -0.1$. (All the quantities are dimensionless.)

new more complete form. The number of variables increases in the process, however, which complicates the integration of the "segment" of the equation under consideration.

2. Let us consider Eq. (14) in a local approximation, retaining only the first five parameters (f_{10}, f_{01}, g_{10}, g_{01}, λ_{00}) and discarding terms containing derivatives with respect to all the parameters, assuming them to be small.

We will have

$$B^2 \frac{d^3\varphi}{d\eta^3} + \left(f_{10}\varphi + \frac{g_{10}}{2}\varphi + \frac{\eta}{2} g_{01} + B\lambda_{00}\right)\frac{d^2\varphi}{d\eta^2} + f_{10}\left[1 - \left(\frac{d\varphi}{d\eta}\right)^2\right] + f_{01}\left(1 - \frac{d\varphi}{d\eta}\right) = 0, \tag{16}$$

$$\varphi = \frac{d\varphi}{d\eta} = 0 \quad \text{at} \quad \eta = 0, \quad \frac{d\varphi}{d\eta} \to 1 \quad \text{as} \quad \eta \to \infty.$$

The results of the integration of this equation can be used in an approximate solution of a particular problem (using some integral equation for the boundary layer) for the case of arbitrary functions U(x, t), z(x, t), and $v_w(x, t)$. Henceforth, for the purpose of reducing the number of parameters in the integration, we will express one of these through the others using the momentum equation, taking the thickness $\delta^{**}(x, t)$ of momentum loss as the scale of the problem. If we neglect derivatives with respect to the parameters, then it is easy to obtain the transformed momentum equation in the presence of suction or injection for $h = \delta^{**}$ in the following form:

$$g_{10} = F(f_{10}, f_{01}, g_{01}, \lambda_{00}) = 2\left[\zeta - f_{10}(2 + H) - \left(f_{01} + \frac{g_{01}}{2}\right) H - \lambda_{00}\right], \tag{17}$$

where

$$H = \frac{1}{B}\int_0^\infty \left(1 - \frac{d\varphi}{d\eta}\right) d\eta.$$

The substitution of Eq. (17) into (16) decreases the number of parameters in the universal equation to four. In this case Eq. (16) changes into the Blasius equation for the boundary layer at a plate if one sets $f_{10} = f_{01} = g_{01} = \lambda_{00} = 0$ and $B = 0.47$. Equation (16) with allowance for (17) was integrated on a BÉSM-2 computer by the trial-run method with iterations for different values of the parameters. The curves obtained, some of which are shown in Fig. 1, indicate that, both in regions of convergent and divergent channels and with acceleration and slowing of the motion with time, suction ($\lambda_{00} > 0$) increases friction but delays the separation of the boundary layer, while injection ($\lambda_{00} < 0$) decreases friction but promotes the onset of separation. The dependence on the other parameters is identical to that which was presented in [3]. The results of the integration make it possible to obtain a linear approximation of the functional F, valid for small values of the parameters, in the form

$$F = a_1 + a_2 f_{10} + a_3 f_{01} + a_4 g_{01} + a_5 \lambda_{00}, \tag{18}$$

where $a_1 = 0.44$; $a_2 = -5.35$; $a_3 = -1.65$; $a_4 = -2.1$; $a_5 = -0.9$.

In the solution of a concrete problem with assigned functions U(x, t) and $v_w(x, t)$ the quantity z is found from Eq. (18), which with allowance for Eq. (10) takes the form

$$Uz' - a_4\dot{z} = \left(a_2U' + a_3\frac{\dot{U}}{U}\right)z - \frac{a_5v_w}{\sqrt{\nu}}\sqrt{z} + a_1.$$

From the equation obtained it is simple to find the values of the thickness of momentum loss and the reduced friction, for example, for an asymptotic profile of suction at a plate. Taking v_w = const and $z' = \dot{z} = U' = \dot{U} = 0$, we find $\delta^{**} = 0.49\,(\nu/-v_w)$ and $\zeta = 0.47$. An exact solution gives

$$\delta^{**} = 0.5\,\frac{\nu}{-v_w} \text{ and } \zeta = 0{,}5.$$

In conclusion, we note that the effect of nonuniformity and of unsteadiness of suction of the liquid in the boundary layer can be brought out in the integration of the universal Eq. (14) if the derivatives with respect to the parameters λ_{10} and λ_{01}, respectively, are retained on the right side of the latter. The discarding of parameters containing one or another higher derivatives of the velocity at the outer limit of the boundary layer and the velocity of suction or injection with respect to the longitudinal coordinate and time, which is necessary in the integration of the universal equation, prevents one from using this method in the case of periodic functions U(x, t) and $v_w(x, t)$ with a high frequency of variation of the velocities.

NOTATION

x, y are the longitudinal and transverse coordinates in boundary layer;
t is the time;
η is the dimensionless transverse coordinate;
U is the velocity at outer limit of boundary layer;
v_w is the velocity of suction or injection;
ψ is the stream function;
φ is the dimensionless stream function;
u, v are the projections of velocity in boundary layer on x and y axes, respectively;
ν is the coefficient of kinematic viscosity;
h is the scale of transverse coordinate in boundary layer;
$z = h^2/\nu$;
F, H are the characteristic functions;
δ^{**} is the thickness of momentum loss;
ζ is the reduced coefficient of friction;
B is the normalizing factor;
f_{kn}, g_{lm}, λ_{ij} are the dimensionless parameters.

LITERATURE CITED

1. L. G. Loitsyanskii, Mechanics of Liquid and Gas [in Russian], Nauka, Moscow (1973).
2. Y. Y. Chan, Amer. Inst. Aeronaut. Astronaut. J., 7, No. 3, 562-563 (1969).
3. O. N. Bushmarin and Yu. V. Saraev, Inzh.-Fiz. Zh., 27, No. 1, 110-118 (1974).